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PARTITIONING GRAPHS OF BOUNDED TREE-WIDTH

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The paper discusses vertex partitions and edge partitions of graphs of bounded tree-width into graphs of smaller tree-width. The first part of the paper proves the existence of several kinds of such partitions. The second part, which has a Ramsey-theoretic character, shows that some of the results of the first part are close to being best possible. The last section of the paper presents a result on partitioning graphs of bounded tree-width into star-forests.

1. Introduction

Graphs in this paper are simple, that is, without loops or multiple edges. In some proofs, it will be convenient to direct the edges of a graph, thereby forming a directed graph. The set of vertices of a graph G will be denoted by $V(G)$, and the set of edges of G will be denoted by $E(G)$. An *edge partition* of a graph G is a set $\{A_1,A_2,\dots,A_k\}$ of subgraphs of G such that $\bigcup_{i=1}^{k} E(A_i)=E(G)$. Similarly, a *vertex partition* of G is a set $\{A_1,A_2,\dots,A_k\}$ of induced subgraphs of G such that $\bigcup_{i=1}^{k} V(A_i)=V(G)$.

A *proper vertex k-coloring* is then a vertex partition into k edgeless graphs. A *proper edge k-coloring* is then an edge partition into k matchings. Of course, there are many results on proper coloring, but other types of partitions have been studied as well.

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Tutte [7] considered edge partitions of arbitrary graphs into planar graphs. Nash-Williams [5] considered edge partitions of arbitrary graphs into forests, while Chartrand and Kronk [2] considered vertex partitions of arbitrary graphs into forests. Further types of partitions can be found in [1]. These results answer questions of the following type: Given classes of graphs $\mathcal{G}$, $\mathcal{G}_1$, $\mathcal{G}_2$, $\ldots$, $\mathcal{G}_k$, does every graph $G \in \mathcal{G}$ have a vertex partition (or an edge partition) $\{G_1, G_2, \ldots, G_k\}$ such that each G_i is in $\mathcal{G}_i$? Note that a negative answer to a question of this type may be viewed as a Ramsey-theoretic result. In this and other papers [3], [4], we address several questions of this type for classes of graphs that generalize the classes of planar graphs, forests, edgeless graphs, and matchings. For example, in [3], we consider partitions of graphs embedded on general surfaces, and in [4], we find partitions into two graphs, each of which has only bounded size components.

Let k be a positive integer. A *k-clique* is a complete graph or subgraph. A *k-tree* is a graph defined inductively as follows: A k-clique is a k-tree. If G is a k-tree, and K is a k-clique of G, then a graph obtained from G by adding a new vertex and joining it by new edges to all vertices of K is a k-tree. Any subgraph of a k-tree is a *partial k-tree.* The *tree-width* of a graph G is zero if G is edgeless; otherwise it is the smallest integer k such that G is a partial k-tree. Nontrivial forests have tree-width 1, while every graph has some tree-width.

In this paper, we consider vertex partitions and edge partitions of partial k-trees. First, we show that a partial (k_1+k_2+1)-tree has a vertex partition $\{G_1, G_2\}$, where, for $i \in \{1,2\}$, G_i is a partial k_i-tree. This implies that a partial $(2k-1)$-tree has a vertex partition into k forests. These results are best possible, in the following sense. We show that, given any partial k_1-tree T_1 and any partial k_2-tree T_2, there is a partial (k_1+k_2)-tree G such that, for every vertex partition $\{G_1, G_2\}$ of G, there is an $i \in \{1,2\}$ such that G_i contains a subgraph isomorphic to T_i. This implies that, given any tree T, there is a partial k-tree G such that, for every vertex partition of G into k graphs, one of these graphs contains a subgraph isomorphic to T.

Similar results are proved for edge partitions. A graph of tree-width at most $k_1 + k_2$ has an edge partition $\{G_1, G_2\}$ where, for $i \in \{1,2\}$, G_i has tree-width at most k_i. This implies that a graph of tree-width at most k has an edge partition into k forests. Different from vertex partitions, there is a tree T' (on 31 vertices) such that every partial k-tree has an edge partition into k graphs, none of which contains a subgraph isomorphic to T'. On the other hand, we conjecture that, given any tree T, there is a partial k-tree G such that, for every edge partition of G into k graphs, one of these graphs contains a subgraph isomorphic to a subdivision of T. We prove this in the case $k = 2$.

2. Partitioning Graphs of Low Tree-Width

In this section, we shall prove two results on partitioning graphs of bounded tree-width into graphs of smaller tree-width.

Theorem 2.1. *Suppose $k=k_1+k_2+1$ for some positive integers k_1 and k_2. Every k-tree has a vertex partition into a k_1-tree and a k_2-tree.*

Proof. We will prove by induction that every k-tree G has a vertex partition $\{G_1,G_2\}$ such that, for $i\in\{1,2\}$, G_i is a k_i-tree, and every k-clique C of G satisfies $k_i\leq|V(C)\cap V(G_i)|\leq k_i+1$. This is clearly true if $|V(G)|=k$; otherwise there is a vertex v whose neighborhood is a k-clique K, such that $G-v$ is a smaller k-tree, and thus has such a vertex partition $\{T_1,T_2\}$. If $|V(K)\cap V(T_1)|=k_1$, then let $P=\{T_1\cup\{v\},T_2\}$, else let $P=\{T_1,T_2\cup\{v\}\}$. In either case P is an appropriate partition of P. ∎

The following is an analogue of Theorem 2.1 for edge partitions.

Theorem 2.2. *Suppose $k=k_1+k_2$ for some positive integers k_1 and k_2. Every k-tree has an edge partition into a k_1-tree and a k_2-tree.*

Proof. We shall describe the required partition by coloring the edges of a k-tree G red and blue. We shall prove by induction on number of vertices that

(1) the red subgraph is a k_1-tree and the blue subgraph is a k_2-tree; and

(2) every k-clique of G can be vertex partitioned into a k_1-clique L_1 and a k_2-clique L_2, such that each edge of L_1 is red and each edge of L_2 is blue.

A desired coloring of a k-clique can be achieved by coloring the edges of a k_1-subclique red and the remaining edges blue. Suppose now that G is a k-tree that is not a k-clique. Then G contains a vertex v whose neighborhood is a k-clique K. We apply the inductive hypothesis to color the edges of $G-v$ so that (1) and (2) hold. It follows that $V(K)$ can be partitioned into V_1 and V_2 with k_1 and k_2 elements, respectively, so that all the edges between vertices in V_1 are red, and all the edges between vertices in V_2 are blue. Color the edges between v and V_1 red, and the edges between v and V_2 blue. It is easy to check that the resulting coloring of the edges of G satisfies both (1) and (2). ∎

Theorems 2.1 and 2.2 have the following immediate corollaries.

Corollary 2.3. *Let k, l and k_1, k_2, ..., k_l be positive integers such that $k=k_1+k_2+\ldots+k_l+l-1$. Then every k-tree has a vertex partition $\{G_1,G_2,\ldots,G_l\}$ where every G_i is a k_i-tree. In particular, every $(2k-1)$-tree has a vertex partition into k trees.*

Corollary 2.4. *Let k, l and k_1, k_2, ..., k_l be positive integers such that $k=k_1+k_2+\ldots+k_l$. Then every k-tree has an edge partition $\{G_1,G_2,\ldots,G_l\}$ where every G_i is a k_i-tree. In particular, every k-tree has an edge partition into k trees.*

3. Large k-Trees in the Parts of the Partitions

Observe that every k-tree except the k-clique contains a $(k+1)$-clique. Thus, it is clear that the result of Theorem 2.1 is the best possible in the sense that if $k=k_1+k_2$, then, for every vertex partition $\{G_1,G_2\}$ of a k-tree G other than the k-clique, the tree-width of G_1 is at least k_1 or the tree-width of G_2 is at least k_2. Theorem 3.2 below strengthens this statement by showing that if G is a large k-tree, then G_i contains a large k_i-tree for at least one i in $\{1,2\}$. Before formally stating this theorem, we need a formal definition of the term "large" in the previous sentence.

Let k be a positive integer. Now we will define some useful k-trees, each of which will have a *level* function λ defined on its vertices. Let the level of a subgraph of a graph with a level function be the maximum level of its vertices. If $()$ represents the empty sequence, let $T(k,())$ be the k-clique, and let each of its vertices have level zero. For a nonnegative integer l, let $\mathbf{r}=(r_1,r_2,\dots,r_l)$ be a sequence of nonnegative integers. We will proceed by induction on l. The k-tree $T(k,\mathbf{r})$ and its level function are obtained from the k-tree $T=T(k,(r_1,r_2,\dots,r_{l-1}))$ (or $T=T(k,())$ if $l=1$) and its level function by the following: For each k-clique K of T that has level $l-1$, add r_l new vertices, join each of them to all the vertices of K, and declare the new vertices to be at level l. Also, for each new vertex v added, let $K(v)$ denote this k-clique K of level $l-1$. If the sequence $\mathbf{r}$ has length l, and all its entries have the same value r, then $T(k,\mathbf{r})$ may be denoted as $T(k,l,r)$.

The following is a simple proposition, whose proof is evident.

Proposition 3.1. *The graph $T(k,\mathbf{r})$ is a k-tree, and every k-tree is a subgraph of $T(k,l,r)$ for some l and r.*

In some arguments, it will be useful to consider a directed version of $T(k,\mathbf{r})$. In each such application, the level-0 vertices will receive an arbitrary linear order, the edges of $T(k,\mathbf{r})$ between level-0 vertices will be directed according to this order (from lower to higher), and each of the remaining edges will be directed from the lower level vertex toward the higher level vertex. For two vertices x and y of $T(k,\mathbf{r})$, we will write $x\preceq y$ if $x=y$, or if there is a directed path from x to y. Note that as $T(k,\mathbf{r})$ has no directed cycles, the relation $\preceq$ is a partial order.

Theorem 3.2. *Let $k=k_1+k_2$ for some positive integers k_1 and k_2, and let l and r be positive integers. There is an integer L such that for every vertex partition $\{G_1,G_2\}$ of $T(k,L,r)$ at least one of the following holds:*

(i) *G_1 contains a subgraph isomorphic to $T(k_1,l,r)$.*

(ii) *G_2 contains a subgraph isomorphic to $T(k_2,l,r)$.*

Proof. Let

$$p=k_1+r\left(\frac{(k_1r)^l-1}{k_1r-1}\right),$$

which is the number of vertices of $T(k_1,l,r)$, and let $L=(k_2+l)p$. Since the relation $\preceq$ is a partial order, the vertices of $T(k_1,l,r)$ can be listed as v_1, v_2, $\ldots$, v_p, so that the first k_1 vertices on the list all have level zero, and $i \leq j$ whenever $v_i \preceq v_j$. For an integer i in $\{1,2,\ldots,p\}$, let H_i denote the subgraph of $T(k_1,l,r)$ induced by $\{v_1,v_2,\ldots,v_i\}$.

We shall prove the following statement, which implies the conclusion of Theorem 3.2 by setting $i=p$.

For every i in $\{1,2,\ldots,p\}$ and every vertex partition $\{G_1^i,G_2^i\}$ of $T(k,(k_2+l)i,r)$ at least one of the following holds.

(1) G_1^i has a subgraph isomorphic to H_i.

(2) G_2^i has a subgraph isomorphic to $T(k_2,l,r)$.

Suppose the disjunction (1) or (2) fails for some i, and let i be the smallest integer for which this occurs. Clearly, $i>1$, as (1) trivially holds for $i=1$. Let $\{G_1^i,G_2^i\}$ be a vertex partition of $T(k,(k_2+l)i,r)$ for which both (1) and (2) fail. Since (2) fails, G_2^i restricted to $T(k,(k_2+l)(i-1),r)$ cannot contain a subgraph isomorphic to $T(k_2,l,r)$. By the minimality of i, G_1^i then has a subgraph H'_{i-1} that is isomorphic to H_{i-1} and contains no vertices of level greater than $(k_2+l)(i-1)$. The order on the vertices of $T(k_1,l,r)$ implies that H'_{i-1} contains a clique K such that adding a new vertex to H'_{i-1} and making it adjacent to all vertices of K would result in a graph isomorphic to H_i.

By construction, $T(k,(k_2+l)i,r)$ has a k-clique K' of level $(k_2+l)(i-1)$ that contains K. Thus $T(k,(k_2+l)i,r)$ has a k-clique K'' that contains K and has a vertex of each level in $\{(k_2+l)(i-1)+1,(k_2+l)(i-1)+2,\ldots,(k_2+l)(i-1)+k-|V(K)|\}$. Let $J_0=V(K'')$, and for $j\in\{1,2,\ldots,l\}$, let J_j be J_{j-1} together with all $v\in T(k,(k_2+l)i,r)\setminus J_{j-1}$ such that v is of level $(k_2+l)(i-1)+k-|V(K)|+j$ and is adjacent to each vertex of K, as well as $k-|V(K)|$ other vertices of J_{j-1}. If every vertex of $J_l\setminus V(K)$ is in G_2^i, then the vertices in this set induce a graph isomorphic to $T(k-|V(K)|,l,r)$, which contains $T(k_2,l,r)$; a contradiction. Thus some vertex $w\in J_l\setminus V(K)$ is in G_1^i. But then $V(H'_{i-1})\cup\{w\}$ induces a graph isomorphic to H_i, as required. ∎

It is immediate that Theorem 3.2 implies the following:

Corollary 3.3. *Let k, l, and r be positive integers. There is an integer L such that, for every vertex partition $\{G_1,G_2,\ldots,G_k\}$ of $T(k,L,r)$, at least one of the G_i's contains a subgraph isomorphic to $T(1,l,r)$.*

We conjecture that a result analogous to Corollary 3.3 can also be proved for edge partitions. More precisely, we propose the following:

Conjecture 3.4. *Let k, l, and r be positive integers. There are integers L and R such that, for every edge partition $\{G_1, G_2, \ldots, G_k\}$ of $T(k,L,R)$, at least one of the G_i's contains a subgraph isomorphic to a subdivision of $T(1,l,r)$.*

We remark that Conjecture 3.4 fails if the phrase "a subdivision of" is removed, as is demonstrated in Theorem 3.9 later in this section.

The following theorem speaks about unavoidable monochromatic subgraphs of a $T(2,L,R)$, for large values of L and R, whose edges have been 2-colored. This theorem may be viewed as supporting evidence for Conjecture 3.4.

Theorem 3.5. *Let l and r be positive integers. There are integers L and R depending only on l and r such that if $T(2,L,R)$ has each of its edges colored either blue or red, then it has a blue subgraph isomorphic to $T(1,l,r)$ or a red subgraph isomorphic to a subdivision of $T(1,l,r)$.*

We postpone the proof of this theorem until after we establish three auxiliary lemmas. The first two of these lemmas, which allow us to derive the general statement of the theorem from a proof for a special case when $R = r = 1$, were discovered during an informal discussion between Bruce Richter, Neil Robertson, Paul Seymour, and one of the authors.

We first add yet more structure to the directed version of $T(k,\mathbf{r})$, where $\mathbf{r}$ has length l. Color the vertices of $T(k,\mathbf{r})$ with colors 0, 1, ..., k by coloring the vertices of the level-0 k-clique in the $\preceq$-ascending order and extending this coloring to the unique proper $(k+1)$-coloring of the entire $T(k,\mathbf{r})$. This is the *canonical vertex coloring* of $T(k,\mathbf{r})$.

Let G be a graph and let J be a subgraph of G. A *bridge* of J in G is a subgraph B of G that satisfies the following:

(1) B is not a subgraph of J.
(2) Every vertex of B that is incident in G with an edge not in B lies in J.
(3) No proper subgraph of B satisfies both (1) and (2).

Since each k-clique K is linearly ordered, it has a unique $\preceq$-maximal vertex, which we denote $\max(K)$. If the level of K is m, where $0 < m < l$, then K has exactly r_{m+1} *upper bridges*, each with vertex set $\{v : w \preceq v\} \cup V(K)$ for some vertex w such that $K(w) = K$, and one *lower bridge*, which uses the remaining vertices. A level-l k-clique has just one bridge, which is the lower bridge; the level-0 k-clique has only upper bridges, and their number is r_1. For $G = T(k,\mathbf{r})$ and a level-m k-clique K, for $m < l$, let $G(\succeq K)$ denote the subgraph of G formed by K and its upper bridges. Observe that $G(\succeq K)$ is isomorphic to $T(k,(r_{m+1}, r_{m+2}, \cdots, r_l))$.

For each vertex v of positive level, the *predecessor* of v, denoted $\mathrm{pre}(v)$, is defined as the vertex $\max(K(v))$. Alternatively, $\mathrm{pre}(v)$ is the $\preceq$-maximal vertex in $\{w : w \preceq v, w \neq v\}$. Note that $\mathrm{pre}(v)$ is adjacent to v and has level one less than v.

For any k-clique K, and any vertex v with $\max(K) \preceq v$, the *canonical path* from $\max(K)$ to v is described as follows. If $\max(K) = v$, this canonical path consists of the single vertex v. If $\max(K) \neq v$, then the canonical path from $\max(K)$ to v consists of the canonical path from $\max(K)$ to $\mathrm{pre}(v)$ extended by the edge $\mathrm{pre}(v)v$.

For a nonnegative integer $m \le l$, define the graph homomorphism $\hom_m$ from $T(k,\mathbf{r})$ to $T(k,(r_1,r_2,\ldots,r_m,1,1,\ldots,1))$ (where there are $l-m$ ones) as follows. For two vertices v and w, $\hom_m(v)=\hom_m(w)$ if and only if $v=w$ or v and w are in the same upper bridge of some level-m k-clique, and the canonical paths from $\max(K)$ to v and from $\max(K)$ to w have identical vertex colorings as given by the canonical vertex coloring. In particular, these canonical paths have the same length, so that v and w have the same level. This homomorphism is unique, up to automorphism.

An edge coloring of $T(k,\mathbf{r})$ is *m-coherent* if and only if $\hom_m(e)=\hom_m(f)$ implies that e and f have the same color. The number of 0-coherent edge colorings of $T(k,\mathbf{r})$ is the same as the number of edge colorings of $T(k,l,1)$. For a positive integer m, the graph $T(k,l,1)$ has k^m level-m edges. Thus $T(k,l,1)$ has $\sum_{m=1}^{l} k^m = k(k^l-1)/(k-1)$ edges not in the level-0 k-clique. These edges can be colored in $c^{k(k^l-1)/(k-1)}$ ways with c colors.

Lemma 3.6. *Let c and k be positive integers, let L be a nonnegative integer, and let $\mathbf{r}$ be a sequence of L positive integers. There is a sequence $\mathbf{s}$ of L positive integers depending only on c, k, L, and $\mathbf{r}$ such that every edge-coloring of $T(k,\mathbf{s})$ with c colors induces a 0-coherent coloring of some subgraph of it isomorphic to $T(k,\mathbf{r})$.*

Proof. Let $c,k,L,\mathbf{r}$ be as in the statement, where $\mathbf{r}=(r_1,r_2,\ldots,r_L)$. For each m in $\{0,1,\ldots,L-1\}$, let $s_{m+1}=(r_{m+1}-1)c^{k(k^{L-m}-1)/(k-1)}+1$. Let $\mathbf{r}_0=\mathbf{r}$. For each m in $\{1,2,\ldots,L-1\}$, let

$$\mathbf{r}_m = (s_1,s_2,\ldots,s_m,r_{m+1},r_{m+2},\ldots,r_L).$$

Let $\mathbf{r}_L=\mathbf{s}=(s_1,s_2,\ldots,s_L)$. Color the edges of $T(k,\mathbf{s})$ with c colors. We shall construct a descending sequence $G_L=T(k,\mathbf{s})$, G_{L-1}, $\ldots$, G_0 of subgraphs of $T(k,\mathbf{s})$ such that, for every integer m in $\{0,1,\ldots,L\}$, the graph G_m is isomorphic to $T(k,\mathbf{r}_m)$, whose coloring is m-coherent.

By assumption, G_L equals $T(k,\mathbf{r}_L)$, and its coloring is L-coherent. Assume now that $m\in\{0,1,\ldots,L-1\}$ and that G_{m+1} is isomorphic to $T(k,\mathbf{r}_{m+1})$, whose coloring is $(m+1)$-coherent. Let $\mathcal{K}$ denote the set of k-cliques of G_{m+1} that have level m. Each $K\in\mathcal{K}$ has s_{m+1} upper bridges and $G_{m+1}(\succeq K)$ is isomorphic to

$$T(k,(s_{m+1},r_{m+2},r_{m+3},\ldots,r_L))$$

whose coloring is 1-coherent. Subject to this coherency condition, each upper bridge can be colored in $c^{k(k^{L-m}-1)/(k-1)}$ ways. By the Pigeon-Hole Principle, there are r_{m+1} identically colored bridges. Let $B(K)$ be the set of vertices in these identically colored bridges. We obtain G_m, colored m-coherently, by deleting from G_{m+1} all vertices of level greater than m which are not in $B(K)$ for some $K\in\mathcal{K}$. The result follows by induction. ∎

Let P be an (undirected) path in G whose vertices, listed in the order in which they appear on P, are v_0, v_1, $\ldots$, v_n. The path P is *strictly l-ascending* if $\lambda(v_0)<\lambda(v_1)<\cdots<\lambda(v_n)$ and $n=l$. The path P is *l-ascending* if there are indices i_0, i_1, $\ldots$, i_l such that $0=i_0<i_1<\cdots<i_l=n$ and $\lambda(v_{i_p})<\lambda(v_q)$ whenever $i_p<q$.

Lemma 3.7. *Let $G = T(k,L,r)$ be 0-coherently edge-colored with red and blue, and let $H = T(k,L,1)$ have the edge-coloring induced by the homomorphism* $\hom_0$ *from G to H.*

(i) *If H has a blue strictly l-ascending path, then G has a blue subgraph isomorphic to $T(1,l,r)$.*

(ii) *If H has a red l-ascending path, then G has a red subgraph isomorphic to a subdivision of $T(1,l,r)$.*

Proof. We shall prove (ii). The proof of (i) is very similar, and is left for the reader. Let P be an l-ascending path in H whose vertices are labeled as in the definition immediately preceding Lemma 3.7. Let K be the level-0 k-clique. Without loss of generality, if $\lambda(v_0)=0$, then we may assume that $v_0 = \max(K)$. We shall construct a sequence of k-cliques $J_0, J_1, \ldots, J_l$ where, for each $p \in \{0,1,\ldots,l\}$, $v_{i_p} = \max(J_p)$ and all the vertices v_s for which $s > i_p$ are in one upper bridge of J_p.

Let $w = \max(K)$. Let Q be the canonical path from w to v_{i_0}, with vertices, in ascending order, $w = w_0, w_1, \ldots, w_m = v_{i_0}$. The following inductive construction defines a sequence of k-cliques $K = K_0, K_1, \ldots, K_m$ such that, for all $i \in \{0,1,\ldots,m\}$, $w_i = \max(K_i)$ and v_n is in an upper bridge of K_i. Suppose $i \in \{0,1,\ldots,m-1\}$, and K_i has been defined. Let K_{i+1} be the k-clique that satisfies $w_{i+1} \in V(K_{i+1}) \subseteq V(K_i) \cup \{w_{i+1}\}$ and v_n is in an upper bridge of $V(K_{i+1})$.

Let $J_0 = K_m$. Now $v_{i_0} = \max(J_0)$ by construction. Moreover, since v_n is in an upper bridge of J_0 and $\lambda(v_{i_0}) < \lambda(v_q)$ for all $q > i_0$, the subpath of P from v_{i_0+1} to v_n is in the same upper bridge of J_0. Similarly to the construction of J_0 from K, construct J_{p+1} from J_p for each $p \in \{0,1,\ldots,l-1\}$ to obtain the required sequence $J_0, J_1, \ldots, J_l$.

For each $p \in \{0,1,\ldots,l\}$, we shall construct a subdivision S_p of $T(1,p,r)$ in G such that $\hom_0$ sends S_p into P (that is, S_p is contained in the preimage of P) and the vertices of S_p corresponding to vertices of level t in $T(1,p,r)$ are precisely the vertices sent by $\hom_0$ to vertex v_{i_t} in H.

Clearly we can let S_0 consist of a single vertex in the preimage of v_{i_0}. Now suppose that we have constructed S_p as required for some $p \in \{0,1,\ldots,l-1\}$. We construct S_{p+1} as follows. Each leaf w of S_p is a preimage of v_{i_p} and there is a preimage W_p (a k-clique) of J_p such that $w = \max(W_p)$. Now W_p has r upper bridges, and in each such bridge we add a path from w to a preimage of $v_{i_{p+1}}$, this part being a preimage of the subpath of P between v_{i_p} and $v_{i_{p+1}}$. ∎

Lemma 3.8. *For a positive integer l, let $L = l^2 + l$. Color each edge of $T(2,L,1)$ either red or blue. Then at least one of the following holds:*

(i) *$T(2,L,1)$ has a blue strictly l-ascending path.*

(ii) *$T(2,L,1)$ has a red l-ascending path.*

Proof. We begin by labeling the vertices of $T(2,L,1)$ inductively as follows: Label the two vertices of level zero by 0 and 1. Assume that l is a positive integer and all vertices of levels lower than l have been labeled. Let v be a vertex of level l. Then

v is adjacent to exactly two vertices labeled u and w whose levels are less than l. Then label v by $\frac{u+w}{2}$. In the remainder of the proof, we shall blur the distinction between a vertex and its label, and we will refer to a vertex labeled x as simply "vertex x". When referring to vertices, we will use binary notation with exponents indicating repeated parts of strings. Thus, for example, the number 0.001010111111 may be written as $0.0(01)^3 1^5$.

Now, embed $T(2,L,1)$ in the plane by placing a vertex x at the coordinates $(x, \lambda(x))$ and drawing edges as straight line segments. See Figure 1. We label the faces of this embedding by giving each finite face the label of the vertex of the highest level incident with this face, and by labeling the infinite face by ∞.

Let $T^*(2,L,1)$ be the plane dual of $T(2,L,1)$. The vertices of $T^*(2,L,1)$ are labeled with the corresponding face labels of $T(2,L,1)$, the faces of $T^*(2,L,1)$ are labeled with the corresponding vertex labels of $T(2,L,1)$, and the edges of $T^*(2,L,1)$ are colored with the corresponding edge colors of $T(2,L,1)$. We shall examine which pairs of vertices of $T(2,L,1)$ are connected by red paths, and this will be accomplished by studying blue circuits in $T^*(2,L,1)$, which correspond to planar edge-cuts in $T(2,L,1)$. The plane embeddings of $T(2,4,1)$ and $T^*(2,4,1) - \infty$ are depicted in Figure 1.

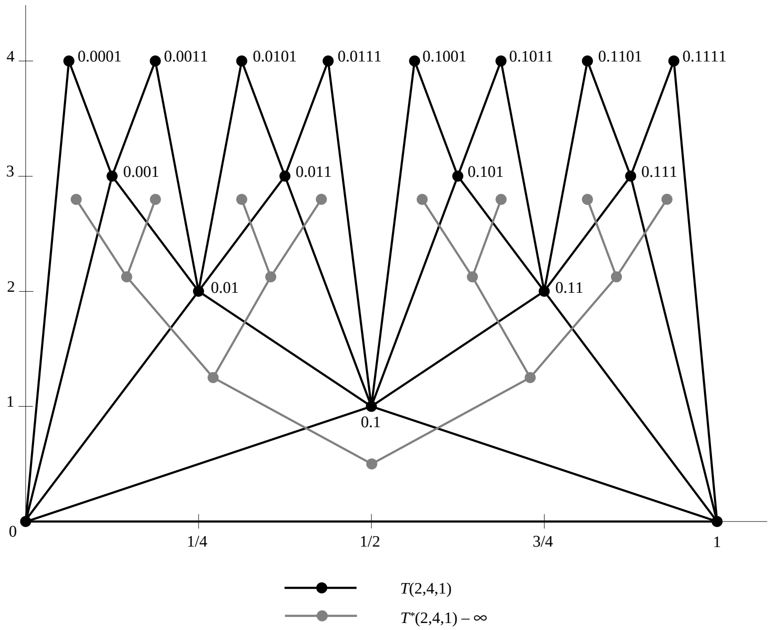

Figure 1

Observe that $T^*(2,L,1) - \infty$ is isomorphic to $T(1, L-1, 2)$, which is a binary tree, with vertices at level n having the form $\frac{m}{2^{n+1}}$ where m is odd. Thus every circuit of $T^*(2,L,1)$ uses vertex ∞. Call a circuit of $T^*(2,L,1)$ an *inner circuit*, or *i-circuit*, if it uses edge $(\frac{1}{2},\infty)$, and call it an *outer circuit*, or *o-circuit*, otherwise. When referring to the level of a vertex of $T^*(2,L,1)-\infty$ we will mean the level of the corresponding vertex of $T(1,L-1,2)$. Each o-circuit C contains a unique vertex v in $V(T^*(2,L,1))\backslash\infty$ whose level λ is minimum. We say that C *bottoms* at level λ.

A blue edge of $T^*(2,L,1)$ is *light blue* if it is in a blue circuit of $T^*(2,L,1)$, and is *dark blue* otherwise. This partition of the blue edges of $T^*(2,L,1)$ induces a partition of blue edges of $T(2,L,1)$.

Suppose first that $T^*(2,L,1)$ has no blue o-circuits that bottom at level $2l-2$ or lower, and no blue i-circuits. Then every pair of vertices of $T(2,L,1)$ whose levels are at most $2l-1$ is connected by a red path. In particular, there is a red path P in $T(2,L,1)$ that connects vertex 0 to vertex $v_l = 0.(01)^{l-1}1$. For each i in $\{1,2,\ldots,l-1\}$, the two-element set of vertices $\{u_i,v_i\} = \{0.(01)^i, 0.(01)^{i-1}1\}$ separates 0 from v_l, and, furthermore, $\lambda(u_i)=2i$ and $\lambda(v_i)=2i-1$. It follows now that P is l-ascending.

We may now assume that $T^*(2,L,1)$ has a blue o-circuit that bottoms at level $2l-2$ or lower, or a blue i-circuit. In either case, $T^*(2,L,1)$ has a light blue path from a vertex whose level is at most $2l-2$ to a leaf of $T^*(2,L,1)-\infty$, whose level is $L-1$. Since $L=l^2+l$, every such path has a subpath with exactly $l(l-1)+1$ edges from a vertex w_0 of level $2l-2$ to that leaf.

An edge uv in $T^*(2,L,1)-\infty$ with $\lambda(u)<\lambda(v)$ is *left* if $u>v$, and *right* if $u<v$. We construct a path P in $T^*(2,L,1)$ inductively as follows: Let P_0 be the light blue path consisting of w_0 only. Suppose that for some $i\in\{0,1,\ldots,l(l-1)\}$, the light blue path P_i has been constructed so that w_0, w_1, $\ldots$, w_i are the consecutive vertices of P_i where $\lambda(w_0)$, $\lambda(w_1)$, $\ldots$, $\lambda(w_i)$ are consecutive integers in increasing order. Let e_i and e_i' be the edges of $T^*(2,L,1)$ that join w_i to vertices of level $\lambda(w_i)+1$. By definition of light blue edges, at least one of e_i and e_i' is light blue. If $i=0$ or only one of e_i and e_i' is light blue, choose w_{i+1} so that the edge w_iw_{i+1} is light blue; otherwise choose w_{i+1} so that the type of the edge w_iw_{i+1}, left or right, is different from the type of $w_{i-1}w_i$. Form P_{i+1} from P_i by appending the edge w_iw_{i+1}. Finally, let $P=P_{l(l-1)+1}$.

Let us examine how the edges of P look in $T(2,L,1)$. If from among two consecutive edges $e=uv$ and $f=vw$ of $T^*(2,L,1)$ with $u\preceq v\preceq w$ one is left and the other is right, then, in $T(2,L,1)$, these edges have form $e=xy$ and $f=yz$ with $x\preceq y\preceq z$, that is, xyz is a strictly 2-ascending path. If, instead, e and f are both left, then in $T(2,L,1)$ these edges have the form $e=xy$ and $f=xz$ with $x\preceq y\preceq z$ and yz being an edge. Similarly, if e and f are both right.

Now, suppose that e_1, e_2, $\ldots$, e_n are edges of P such that, for every i in $\{1,2,\ldots,n-1\}$, the type of e_i differs from the type of e_{i+1} as well as from the type of all edges of P that lie between e_i and e_{i+1}. Then, in $T(2,L,1)$, the edges e_1, e_2, $\ldots$, e_n form a strictly n-ascending path. Thus, if $n\geq l$, then $T(2,L,1)$ has a blue strictly l-ascending path, as required.

From the definition of L, we may now assume that P contains $l+1$ consecutive edges e_0, e_1, …, e_l in order of increasing level and are all left or are all right. Then, in $T(2,L,1)$, these edges have form $e_i = xy_i$ with $x \preceq y_i$ and y_0, y_1, …, y_l being consecutive vertices of a strictly l-ascending path. Since, in constructing P, we switched, whenever possible, between left and right edges, the faces y_0, y_1, …, y_l in $T^*(2,L,1)$ are not separated by a blue circuit of $T^*(2,L,1)$, and, consequently, there is a red path Q in $T(2,L,1)$ joining y_0 to y_l. Note that Q avoids x as y_0 and x are endvertices of a light blue edge, and thus, in $T^*(2,L,1)$, the faces x and y_0 lie on opposite sides of a blue circuit. Furthermore, for each i in $\{1,2,\dots,l-1\}$, the set $\{x,y_i\}$ is a vertex cut in $T(2,L,1)$ that separates the elements of $\{w_0,w_1,\dots,w_{i-1}\}$ from the elements of $\{w_{i+1},w_{i+2},\dots,w_l\}$. Thus Q goes through y_1, y_2, …, y_{l-1} in this order, and hence is a red l-ascending path. ∎

Now we are ready to prove Theorem 3.5.

Proof. Let $\mathbf{r}$ be a sequence of l integers, each of which equals r. Let $L = l^2 + 1$. Let $\mathbf{s}$ be as in Lemma 3.6, and let R be the maximum of the elements of $\mathbf{s}$. Color each of the edges of $T(2,L,R)$ either red or blue. By Lemma 3.6, $T(2,L,R)$ has a subgraph G that is isomorphic to $T(2,L,r)$ and whose edge-coloring is 0-coherent. Consider $T(2,L,1)$ with edge-coloring induced by the canonical homomorphism from G to $T(2,L,1)$. By Lemma 3.8, $T(2,L,1)$ has a blue strictly l-ascending path or a red l-ascending path. Now, Lemma 3.7 implies that in the former case G has blue subgraph isomorphic to $T(1,l,r)$, and in the latter case G has a red subgraph isomorphic to a subdivision of $T(1,l,r)$. The conclusion follows. ∎

The following theorem shows that Conjecture 3.4 fails if the phrase "a subdivision of" is deleted from its statement.

Theorem 3.9. *Every k-tree can be edge partitioned into k graphs none of which contains a subgraph isomorphic to a $T(1,4,2)$.*

Proof. By Proposition 3.1, it suffices to prove the theorem for the k-tree $T(k,l,r)$. We give $T(k,l,r)$ the canonical vertex coloring with colors 0, 1, …, k as described earlier. We color edges of $T(k,l,r)$ with colors 1, 2, …, k as follows. Let $c(v)$ denote the color of vertex v, and let $c(e)$ denote the color of edge e. For an edge e with endvertices u and v with $u \preceq v$, let $c(e) = c(u)$ whenever $c(u) \neq 0$, otherwise let $c(e)$ be a color other than $c(v)$.

We shall show that G has no monochromatic subgraph isomorphic to $T(1,4,2)$. By symmetry, we need only consider color 1. Let G_1 denote the subgraph of G induced by the edges colored 1. Let $d_1(v)$ denote the degree of v in G_1. It suffices to show that there is no path with vertices (in order) w, x, y, z in G_1 for which all $d_1(w)$, $d_1(x)$, $d_1(y)$, and $d_1(z)$ exceed two. Observe that if $c(v) \notin \{0,1\}$, then $d_1(v) \leq 2$. Also, if an edge e has endvertices u and v with $u \preceq v$, $c(u) = 0$, and $c(v) = 1$, then $c(e) \neq 1$. Without loss of generality, $c(w) = c(y) = 0$, $c(x) = c(z) = 1$, $x \preceq y$, and $z \preceq y$. But then there is an edge, namely xz, with both endvertices having the same color; a contradiction. ∎

4. Partitioning into Star-Forests

The last theorem of the paper will be concerned with the issue of partitioning graphs of bounded tree-width into *star-forests*, i.e. forests with no path of more than two edges.

Theorem 4.1. *Every k-tree can be edge partitioned into $k+1$ star-forests.*

Proof. Color the vertices of G as in the proof of Theorem 3.9. Then color each edge with the color of its initial vertex. To see that each monochromatic component is a star, we need only observe that for any two directed edges of the form (u,w) and (v,w) the graph G also contains the edge between u and v, and thus (u,w) and (v,w) have different colors. ∎

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